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Equilibrium characteristics of nearly normal turbulence

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We discuss some consequences of assuming that two different non-linear model equations, and real turbulence are nearly Gaussian. It is supposed when necessary that the process is driven and it is supposed that the processes have become statistically stationary. These problems are discussed from the viewpoint of the Wiener–Hermite expansion for non-linear, nearly Gaussian processes. Expected equilibria forms are related to corresponding expressions obtained from the zero-fourth-cumulant assumption. The spectrum for Burgers' model and for incompressible fluid flow problems is found from this viewpoint to be $E \sim k^{-2}$. The kinematical properties leading to such spectra are discussed. It is noted, as has been remarked earlier, that this spectrum is characteristic of flows with near discontinuities. A conjecture is offered concerning how these discontinuities are related to Gaussianity.

1. Introduction

We shall discuss some consequences of assuming that: (1) model turbulence, and (2) real turbulence is nearly Gaussian and in some form of equilibrium. We shall concentrate on two model equations: (1*a*) the three-mode model, (1*b*) the Burgers' model equation. The real fluid Navier–Stokes incompressible equations will be considered, the order of presentation being (1*a*), (2) and (1*b*). The Cameron–Martin–Wiener representation (sometimes called the Wiener or Wiener–Hermite representation) will be briefly reviewed. We follow a parallel treatment for each of these three sets of equations. It will be found, first of all, that the energy equation in each one of the three cases involves a transfer term which is non-zero only when the process has non-Gaussian parts, as is well known. Then, we take the derivative of the energy equation in each one of these cases and note that by using the equations of motion in the evaluation of the derivative of the transfer function, we obtain a fourth-order moment in each case. If we now suppose that the process is nearly normal, we can replace the fourth moment by products of the second moments following a treatment very similar to that of a zero-fourth-cumulant approximation. In the past this approximation was used to evaluate the time development of processes. When the initial state of the system is far from its equilibrium form, it has been found that such a truncation is inadequate. However, here we shall look for equilibrium forms, for example, forms such that the time derivative of the transfer function vanishes. When we do so we find the known condition for equilibrium for the three-mode model; secondly, the known equilibrium spectrum for Burgers' model, i.e.

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12-2

$E(k)$ proportional to k^{-2}. Finally, for the Navier–Stokes equation proceeding in the way just described, we find an equilibrium spectrum $E(k) \sim k^{-2}$ also (as has been shown previously). We discuss the kinematical characteristics of processes involving real fluids with such spectra. In particular it is noted, as observed much earlier by Townsend, that some flows with near discontinuities in the velocity functions have spectra like k^{-2}.

In most practical problems involving turbulence, one is faced with statistically inhomogeneous turbulence, which is frequently driven and is approximately in statistical equilibrium. The driving source is very often found in large-scale effects deriving their energy from the mean flow. For example, we point to the existence of turbulence in the lower atmosphere which is driven by wind shear in that region. Another important application lies in problems involving turbulent wakes. In such applications we find the turbulence driven by large eddies which couple into the mean shear flow and derive their energy from that flow.

We shall here replace the large-scale energy sources by an equivalent forcing term both in model and real turbulence. We suppose that the process has continued long enough so that it is statistically stationary (in a statistically inhomogeneous problem the forcing term may vary with position). It is easy to generalize the random process expansion for such problems (see Meecham & Jeng 1968, §II). Recently, Saffman (1968) has used this kind of stationary expansion to deal with turbulent diffusion, and has obtained some very promising results in this way.

2. Review of Cameron–Martin–Wiener functionals

The functionals provide a useful representation for nearly Gaussian processes. The use of these functionals in the examination of non-linear stochastic problems is presented more fully elsewhere (see Wiener 1958; Cameron & Martin 1947; Imamura, Meecham & Siegel 1965). It will be useful, however, to review some of the salient characteristics of such expansions here. For this purpose we use only scalar functions of a scalar variable x. Generalizations are given in the references for vector functions of vector arguments. The expansion is based on $a(x)$, the white noise process. The a represents a set of functions with the properties

$$\langle a(x)\rangle = 0, \tag{2.1}$$

$$\langle a(x_1)\, a(x_2)\rangle = \delta(x_1 - x_2), \tag{2.2}$$

plus further moment equations expressing the condition that a be Gaussian. The process is statistically independent from one point, that is from one value of x, to another. The singular set of functions $a(x)$ can be used to represent a random function $u(x)$ when integrated with a suitable regular, non-random weighting function $K(x)$.

$$u(x) = \int K(x - x')\, a(x')\, dx'. \tag{2.3}$$

Since a is Gaussian and independent from point to point, u will be Gaussian at all points. Further, the use of a difference argument guarantees that u will be statistically homogeneous. To represent non-Gaussian characteristics of u one uses polynomial combinations of the ideal random function. They are constructed in such a manner that they are statistically orthogonal. For the first two such functionals we have,

$$\left.\begin{aligned} H^{(0)}(x) &= 1, \quad H^{(1)}(x) = a(x). \\ H^{(2)}(x_1, x_2) &= a(x_1)\, a(x_2) - \delta(x_1 - x_2), \\ H^{(3)}(x_1, x_2, x_3) &= a(x_1)\, a(x_2)\, a(x_3) - a(x_1)\, \delta(x_2 - x_3) - a(x_2)\, \delta(x_3 - x_1) \\ &\quad - a(x_3)\, \delta(x_1 - x_2). \end{aligned}\right\} \tag{2.4}$$

We see that the $H^{(i)}$ are symmetric in their multiple arguments and using (2.1) and (2.2) and corresponding higher moment relations,

$$\langle H^{(i)} H^{(j)} \rangle = 0 \quad (i \neq j); \tag{2.5}$$

also

$$\left.\begin{aligned} \langle H^{(0)} H^{(0)} \rangle &= 1, \\ \langle H^{(1)}(x_1) H^{(1)}(x_2) \rangle &= \delta(x_1 - x_2), \\ \langle H^{(2)}(x_1, x_2) H^{(2)}(x_3, x_4) \rangle &= \delta(x_1 - x_3)\, \delta(x_2 - x_4) + \delta(x_1 - x_4)\, \delta(x_2 - x_3). \end{aligned}\right\} \tag{2.6}$$

Expansions of random functions in terms of these functionals can be made (and can be shown to be complete). Consider a random function $u(x)$. We write, including second-order terms,

$$u(x) = \int K^{(1)}(x - x_1)\, H^{(1)}(x_1)\, dx_1 + \iint K^{(2)}(x - x_2, x - x_3)\, H^{(2)}(x_2, x_3)\, dx_2 dx_3, \tag{2.7}$$

where we have to determine the non-random kernels $K^{(i)}$ using a dynamic equation obeyed by u. We note, because of the symmetry of $H^{(i)}$, that the $K^{(i)}$ can be assumed to be symmetric in their arguments without loss of generality.

3. The three-mode model

Kraichnan (1963) has discussed a three-mode model [also considered by Orszag & Bissonnette (1967)]. The model is one involving three discrete processes as follows:

$$\frac{dx_i(t)}{dt} = A_i x_j(t)\, x_k(t), \tag{3.1}$$

where i, j, k take on the values 1, 2, 3 and are distinct, and where the A_i are constants whose sum is zero. It is known that this system possesses an equilibrium solution such that each random variable x_i is Gaussian at a given time and independent of the other two random variables. The joint characteristics in time are not Gaussian. The method for treating equilibrium, nearly Gaussian processes can be demonstrated for this simple model. We assume the process has become statistically stationary. Multiply (3.1) by $x_l(t)$ and average to find,

$$(d\langle x_i x_l \rangle / dt) = A_i \langle x_l x_j x_k \rangle + A_l \langle x_i x_m x_n \rangle, \tag{3.2}$$

with l, m and n equal to 1, 2, 3 and distinct. If the x_i are now nearly Gaussian, the right side is small, and if exactly Gaussian the moment has zero derivative, as it must if the process is stationary. Differentiate again and substitute for derivatives in the right side using (3.1),

$$(d^2/dt^2)\langle x_i x_l\rangle = A_i\{A_l\langle x_m x_n x_j x_k\rangle + A_j\langle x_l x_k x_i x_k\rangle + A_k\langle x_l x_j x_i x_j\rangle\}$$
$$+ A_l\{A_i\langle x_j x_k x_m x_n\rangle + A_m\langle x_i x_n x_l x_n\rangle + A_n\langle x_i x_m x_l x_m\rangle\}. \quad (3.3)$$

Suppose that (3.3) vanishes (because of stationarity) and that x_i are Gaussian. A solution then is

$$\langle x_i x_l\rangle = \langle x_i^2\rangle\,\delta_{il} \quad (3.4)$$

together with the condition

$$A_i\langle x_j^2\rangle\langle x_k^2\rangle + A_j\langle x_k^2\rangle\langle x_i^2\rangle + A_k\langle x_i^2\rangle\langle x_j^2\rangle = 0.$$

(From (3.4) we conclude that x_i are statistically independent of one another.) These are the known conditions for stationarity (see Orszag & Bissonnette 1967). Further, this process, x_i, gives all moments stationary as said, and is the exact stationary solution.

This stationary problem has been solved by Doi & Imamura (1969) using a time-dependent Wiener process. A brief discussion of this solution will be helpful in elucidating points of importance in more complicated problems later. The fact that while the Wiener representation of the type discussed here is complete for quite general random processes, it is not unique. Discussions of appropriate so-called 'measure preserving transformations' on the representation have been presented elsewhere (see Meecham 1969). It is possible to represent a Gaussian process either by a single term in the expansion of type (2.7) or by a very large number of terms. The process described above, which is a stationary solution of (3.1), in fact is an example of a time-transforming process. The x_i are Gaussian at any given time, but if they are expanded at later times in terms of the white noise process appropriate at the given time, one finds that the series of terms becomes progressively more slowly converging. Alternatively, one may use a time-transforming white noise process in such a way that the x_i are always represented by a single term in the stochastic expansion. Doi & Imamura (1969) have solved the problem represented by (3.1) in this way. They find

$$x_i(t) = P_i(t)H_i(t), \quad (3.5)$$

where the H_i functions of time are given by

$$\dot{H}_i(t) = A_i P_i^{-1} P_j P_k H_j(t) H_k(t), \quad (3.6)$$

with

$$P_i^2 \equiv \langle x_i^2\rangle. \quad (3.7)$$

It is remembered that there is no summation convention in these expressions.

In this solution P_i are constant. The H_i here consists of three discrete, independent Gaussian processes with unit variance. They are analogous to the slightly more general white noise processes presented in (2.4). It is important that the process $H_i(t)$ remain a white-noise process. The conditions necessary to guarantee that this be so amount to requiring that the time derivatives of all moments of H_i vanish. These conditions are discussed in detail by Doi & Imamura (1969). The time correlation, even for this simple discrete random process problem, presents some difficulty. Thus it would be necessary to integrate an ensemble of H_i's using (3.6) in order to obtain these correlations. It happens that the system of equations (3.1), or equivalently (3.6), possesses a closed-form solution so that it is possible to obtain the time characteristics in closed form. It is noted that the solution (3.6) is quite similar to the equations of motion themselves, (3.1). There is, however, the important simplification that in (3.6), the statistics of the random process are known (that is Gaussian), whereas the process in (3.1) has much more general statistical characteristics in the general case.

4. Stationary, driven turbulence

We develop the equations for stationary driven turbulence in an incompressible fluid, relying on previous work for some of the details (see Meecham & Jeng 1968). Define the Fourier transform of the velocity field

$$\mathbf{u}(\mathbf{k},t) = \int e^{i\mathbf{k}\cdot\mathbf{r}}\,\mathbf{u}(\mathbf{r},t)\,d\mathbf{r}$$

(and similarly for $\mathbf{f}$ below). The Fourier transform of the incompressible Navier–Stokes equation is (using the summation convention and with time-dependence implicit)

$$([\partial/\partial t] + \nu k^2)\,u_i(\mathbf{k}) = (i/2)\,(2\pi)^{-3}P_{ijl}(\mathbf{k})\int u_j(\mathbf{k}-\mathbf{k}')\,u_l(\mathbf{k}')\,d\mathbf{k}' + f_i(\mathbf{k}), \tag{4.1}$$

with $$k_i f_i = k_i u_i = 0$$

and $$P_{ijl}(\mathbf{k}) = k_l P_{ij}(\mathbf{k}) + k_j P_{il}(\mathbf{k})$$

and $$P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2.$$

The given (solenoidal) forcing term $\mathbf{f}$ will be assumed to be statistically stationary and homogeneous, Gaussian, and confined to large-scale effects. That is, it is supposed that $\mathbf{f}$ vanishes for k larger than an energy-containing wave-number k_0. The first two random processes in the Cameron–Martin–Wiener representation are (analogous to (2.4))

$$H_i^{(1)}(\mathbf{r}),$$

$$H_{ij}^{(2)}(\mathbf{r}_1,\mathbf{r}_2) = H_i^{(1)}(\mathbf{r}_1)\,H_j^{(1)}(\mathbf{r}_2) - \delta_{ij}\delta(\mathbf{r}_1-\mathbf{r}_2), \tag{4.2}$$

with the covariance property

$$\langle H_i^{(1)}(\mathbf{r}_1)\,H_j^{(1)}(\mathbf{r}_2)\rangle = \delta_{ij}\delta(\mathbf{r}_1-\mathbf{r}_2).$$

We suppose that the turbulence is statistically stationary [it is known that certain necessary conditions must be imposed for stationarity, see Saffman (1968)]; and that it is (at least locally) statistically homogeneous and isotropic. We suppose convergence sufficiently rapid so that the random field can be adequately represented by just the first two terms in the nearly normal expansion at a given instant.

There is considerable experimental evidence that fully developed (decaying) turbulence is nearly normal in many important characteristics; in particular the even moments are related to one another as they would be for a Gaussian process. In figure 1 the comparison of some measured fourth-order moments, with what they would have been had the process been Gaussian, are presented [taken from the work of Frenkiel & Klebanoff (1967*a*)]. Those authors present considerably more data than is shown here, all leading to the conclusion that the even moments are related (to within experimental error) as they would have been had the process been Gaussian. Odd moments are considerably smaller, of order a few per cent in relative value [see, for instance, Frenkiel & Klebanoff (1967*b*)]. Frenkiel & Klebanoff also show that a *single* lowest order term in a kind of generalized Gram–Charlier expansion is sufficient to obtain higher order

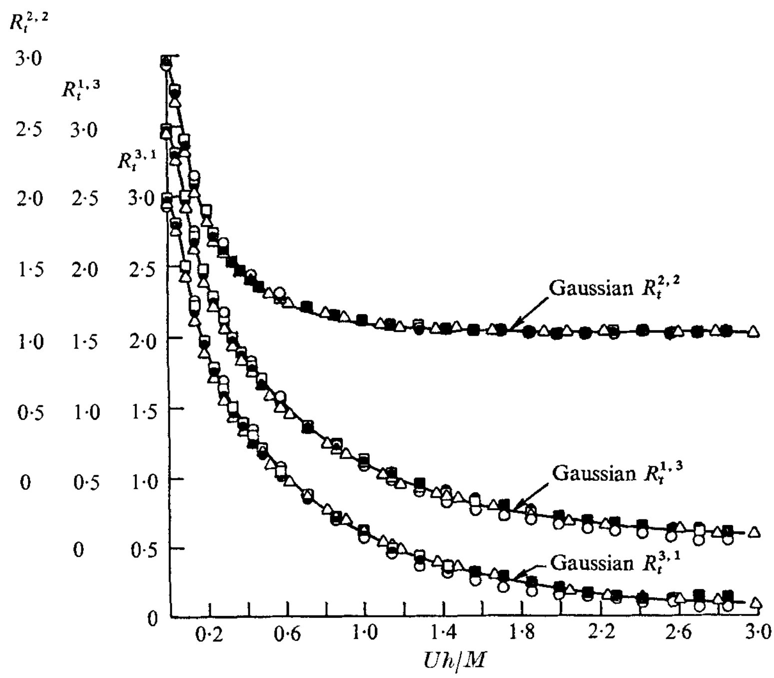

FIGURE 1. Measured correlation coefficients of fourth order compared with the assumption of Gaussian probability density distribution of turbulent velocities (from Frenkiel & Klebanoff 1967*a*).

odd moments from lower order odd moments. This must be taken as striking evidence of near-Gaussianity. Single point probability distributions of fully developed turbulence are known to be very nearly Gaussian as well [see, for instance, Batchelor (1953)]. Further, extensive data supporting the assumption that the turbulent process is nearly normal in certain important characteristics have been presented by Van Atta, Yen & Yeh (1970). On the basis of these data, we assume there is a Wiener representation such that the higher order terms for the velocity fluctuation are smaller than the first term. Continuing with the representation we have for the velocity field,

$$u_i(\mathbf{r},t) \equiv u_i^{(1)} + u_i^{(2)} = \int K_{i\alpha}^{(1)}(\mathbf{r}-\mathbf{r}_1)\, H_\alpha^{(1)}(\mathbf{r}_1)\, d\mathbf{r}_1 + \iint K_{i\alpha\beta}^{(2)}(\mathbf{r}-\mathbf{r}_1, \mathbf{r}-\mathbf{r}_2) H_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2)\, d\mathbf{r}_1 d\mathbf{r}_2, \qquad (4.3)$$

$$f_i(\mathbf{r},t) = \int F^{(1)}_{i\alpha}(\mathbf{r}-\mathbf{r}_1)\,H^{(1)}_{\alpha}(\mathbf{r}_1)\,d\mathbf{r}_1 + \iint F^{(2)}_{i\alpha\beta}(\mathbf{r}-\mathbf{r}_1,\mathbf{r}-\mathbf{r}_2)H_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2)\,d\mathbf{r}_1 d\mathbf{r}_2. \tag{4.4}$$

We suppose that our representation is such that we can use the manifestly statistically homogeneous form, involving kernels that are spatial functions of the difference variable; as stated we assume second-order terms are small.

Returning to (4.1) we construct the usual energy equation. Multiply by the velocity at a second wave-number (and the same time), symmetrize and average to find,

$$[(\partial/\partial t)+\nu(k_1^2+k_2^2)]\langle u_i(\mathbf{k}_1)u_j(\mathbf{k}_2)\rangle = \tfrac{1}{2}i(2\pi)^{-3}P_{i\alpha\beta}(\mathbf{k}_1) \times \int\langle u_\alpha(\mathbf{k}_1-\mathbf{k}')u_\beta(\mathbf{k}')u_j(\mathbf{k}_2)\rangle d\mathbf{k}' + \langle f_i(\mathbf{k}_1)u_j(\mathbf{k}_2)\rangle + \text{same, interchange subscripts } i, j \text{ and variables } \mathbf{k}_1, \mathbf{k}_2. \tag{4.5}$$

For homogeneous processes we know that

$$\langle u_i(\mathbf{k}_1)u_j(\mathbf{k}_2)\rangle \sim V, \quad \mathbf{k}_1+\mathbf{k}_2 = 0; \qquad \sim 0, \quad \text{otherwise}, \tag{4.6}$$

where V is the volume of the turbulence; similar relations hold for other moments.

We follow Proudman & Reid (1954) closely in what follows and find

$$E(k) = [(2\pi)^2 V]^{-1}k^2\langle u_i(\mathbf{k})u_i^*(\mathbf{k})\rangle, \tag{4.7}$$

and for isotropic processes (4.5) becomes

$$([\partial/\partial t]+2\nu k^2)E(k) = T(k)+S(k), \tag{4.8}$$

where

$$T(k) = (i/2V)(2\pi)^{-5}k^2P_{i\alpha\beta}(\mathbf{k})\int\langle u_\alpha(\mathbf{k}-\mathbf{k}')u_\beta(\mathbf{k}')u_i(-\mathbf{k})\rangle d\mathbf{k}' + \text{c.c.}, \tag{4.9}$$

$$S(k) = [8\pi^2 V]^{-1}k^2[\langle f_i(\mathbf{k})u_i^*(\mathbf{k})\rangle + \text{c.c.}]. \tag{4.10}$$

Here c.c. stands for the complex conjugate of a preceding term.

From the properties of the representation (4.3) we know that to lowest order in the non-Gaussian part of the process triple moments like those in (4.9) are proportional to $\mathbf{u}^{(2)}$. Using stationarity we see from (4.8)

$$T(k) = -S(k)+2\nu k^2E(k). \tag{4.11}$$

Consider the time derivative of the third-order moment. Take the derivative of (4.9), use (4.1) and statistical isotropy to find

$$\dot{T}(k) = (\tfrac{1}{4}\pi)k^2\int(Q/k^2k'^2k''^2)\{2[k^2(k''^2-3k'^2)-(k''^2-k'^2)^2]\,\dot{\Phi}(k,k',k'') + 2k''^2(k''^2-k^2)\,\dot{\Phi}(k',k'',k) - k''^2Q\dot{\Psi}(k,k',k'')\}\,d\mathbf{k}', \tag{4.12}$$

where Φ and Ψ are generating scalars for the triple velocity correlation at three points (see Proudman & Reid 1954). For (4.12) we have the additional definitions

$$Q = k^4+k'^4+k''^4-2k^2k'^2-2k'^2k''^2-2k''^2k^2 \tag{4.13}$$

and

$$\mathbf{k}+\mathbf{k}'+\mathbf{k}'' = 0. \tag{4.14}$$

To this point we have made no truncation assumption.

To find the triple correlation scalars Φ and Ψ we would construct from (4.1) the equation relating third- to fourth-order moments. If we again suppose that second-order (non-Gaussian) terms are small, we find essentially the zero-fourth-cumulant-result of Proudman & Reid,

$$\left.\begin{aligned}\dot{\Phi} &\cong \frac{E(k'')}{16\pi^2 k''^2}\left[\frac{E(k')}{k'^2}-\frac{E(k)}{k^2}\right]-\nu(k^2+k'^2+k''^2)\Phi+\Phi_s,\\ \dot{\Psi} &\cong -\nu(k^2+k'^2+k''^2)\Psi+\Psi_s.\end{aligned}\right\} \tag{4.15}$$

The approximation sign stands for the assumption that the fourth-order moments needed here are approximately what they would be for a Gaussian process. Here, Φ_s and Ψ_s are generating scalars for the triple correlation of two velocities with the force **f**. We shall not need these quantities in detail, in the discussion, and merely note that to lowest order they are proportional to the non-Gaussian part of **u** or **f** and vanish for k out of the energy range.

Now suppose we substitute (4.15) in (4.12). For large Reynolds numbers and for k large enough to be out of the energy range (so the source terms vanish) we have for a nearly Gaussian, stationary process

$$\begin{aligned}0 = (\tfrac{1}{4}\pi)k^2\int \frac{Q}{k^2k'^2k''^2}\{2[k^2(k''^2-3k'^2)-(k''^2-k'^2)^2]\,\dot{\Phi}(k,k',k'')\\ +2k''^2(k''^2-k^2)\,\dot{\Phi}(k',k'',k)\}d\mathbf{k}',\end{aligned} \tag{4.16}$$

with

$$\dot{\Phi}(k,k',k'') \cong \frac{E(k'')}{16\pi^2k''^2}\left[\frac{E(k')}{k'^2}-\frac{E(k)}{k^2}\right]. \tag{4.17}$$

For all values of k this is known (see Tatsumi 1960) to have the singular solution $E \sim k^{-2}$. The sum of integrals, (4.16), is convergent for this spectrum. In the physical case, this spectrum would be cut off at the low wave-number end.

It is known that for a one-dimensional process k^{-2} is the high-frequency spectrum which is characteristic of near-discontinuities in the flow. It is easy to show that this is also the case for some three-dimensional flows. Real turbulent flows certainly exhibit near-discontinuities and might be supposed to have an energy spectrum range with a k^{-2} behaviour. Indeed early measurements showed approximately this behaviour (see Dryden 1938).

It is known that Gaussian processes do not have discontinuities of this type. One plausible view might be that the dynamics develop slip discontinuities; such discontinuities require that the different Fourier components have phase coherence. This coherence is then lost as the flow proceeds, leaving a k^{-2} spectrum.

Previous work with the zero-fourth-cumulant approximation involving the integration of (4.8), (4.12) and (4.15) without the source terms, has produced poor results (see Ogura 1963). Concerning this, first, the initial spectra (exponential forms were used) were probably far from equilibrium values and resulted in the process becoming far from Gaussian during the integration time. Thus the central approximation (4.15) was violated. Furthermore, we saw in § 3 that even for a process which is nearly Gaussian, we may expect that a given initial representation will not remain rapidly convergent for much later times. At least we saw in § 3 that this was the case for the simplified three-mode problem. We may reasonably expect a similar behaviour in the more complicated fluid problem discussed here. Thus it is necessary in order to retain rapid convergence, that a time-dependent noise process be used. Although the approximation (4.15) may be adequate for nearly Gaussian processes initially, still if that equation is integrated the implication is that the function Φ is represented adequately at later times by the initial representation. This will often not be the case and consequently the approximation (4.15) will not be fulfilled even for processes which remain nearly Gaussian.

Doi & Imamura (1969) use a time-dependent Wiener process to solve a (non-forced inviscid turbulence) problem of the general type discussed here. They find a solution analogous to that discussed for the three-mode problem in §3. The solution is the familiar equi-partition one, wherein $E \sim k^2$. This process is a (singular) exact solution of the inviscid equations of motion for non-forced turbulence, and invokes a Wiener process which transforms in time in a way analogous to that for the three-mode problem (see (3.6)).

5. Stationary, driven Burgers's model turbulence

Burgers' model will be treated here in a way analogous to that used in §4 for the incompressible Navier–Stokes equation (see Meecham & Su 1969). The transform used is (time again implicit)

$$u(k) = \int_{-\infty}^{\infty} e^{ikx} u(x)\, dx, \tag{5.1}$$

and the same for the forcing term.

The Burgers' equation is

$$\frac{\partial}{\partial t} u(x) + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} + f(x), \tag{5.2}$$

with the transform

$$\dot{u}(k) = -\frac{ik}{4\pi} \int u(k') u(k-k')\, dk' - \nu k^2 u(k) + f(k). \tag{5.3}$$

The energy equation is constructed as before, with analogous definitions

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) E(k) = T(k) + S(k), \tag{5.4}$$

with (see Reid 1956)
$$T(k) = k \int \Phi(k, k')\, dk', \tag{5.5}$$

where Φ is a transform of the triple velocity correlation.

Numerical experiments show that when the forcing term is Gaussian even moments are related as for a Gaussian process. In figure 2 the results of such an experiment are shown (Jeng 1969). We obtain from the zero-fourth-cumulant (sometimes called quasi-normal) approximation

$$\dot{\Phi} \simeq kE(k')E(k'') + k'E(k'')E(k) + k''E(k)E(k') - \nu(k^2 + k'^2 + k''^2)\Phi + \Phi_s, \tag{5.6}$$

with
$$k + k' + k'' = 0.$$

Reasoning as in §4 we have in place of (4.16) and (4.17)

$$-2k^2 E(k) \int_{-\infty}^{\infty} E(k')\, dk' + k^2 \int_{-\infty}^{\infty} E(k+k') E(k')\, dk' \simeq 0. \tag{5.7}$$

This has the singular solution $E \sim k^{-2}$. As remarked earlier this spectrum is characteristic of a velocity field with jumps (shocks) and is known to be the correct result for the large wave-number part of the energy spectrum for this problem (see Jeng 1969). Remarks in §4 about discontinuities and Gaussianity apply here as well.

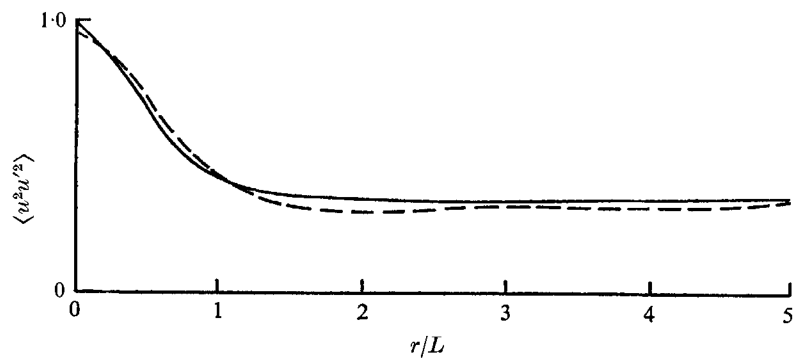

FIGURE 2. The figure shows a numerical experiment for Burgers' model turbulence. The process is driven by a Gaussian forcing term and is at equilibrium. The measured fourth-order velocity moment is compared with what it would be for a Gaussian process. Here L is the scale for the force (from Jeng 1969). $\langle u^2u'^2 \rangle$: ---, measured directly; ——, using quasi-normal assumption. $Re = 50$, 100 realizations.

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